**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Variations calculus

## Lecture 8. Variational problems with isoperimetric conditions

We have the method of the analysis for the problem of the minimization of the integral functional, which depends from one or many unknown functions of one or many variables. The functional can depend from derivatives of unknown functions of arbitrary order. We considered the problems with additional constraints on the boundary of the given set. However, there exist many practical problems with additional constraints for the value of unknown functions at the interior points of the functions domain. We will consider now the variational problem with integral constraints. The necessary conditions of optimality will be obtained by means of Lagrange multipliers method. A spectrum problem will be considered as an example.

**8.1. Problems with isoperimetric conditions**

Consider the functional

 (8.1)

where the smooth enough function *F* is given. Besides, we have the boundary conditions

 (8.2)

and additional equality

 (8.3)

It is called the *isoperimetric condition*. The smooth enough function *G* is given here.

**Problem 8.1**. *Find the function v*, *which satisfies the boundary conditions* (8.2), *the isoperimetric condition* (8.3), *and minimizes the functional I.*

Try to use the known variational method for solving this problem.

**8.2. Begin of solving the problem**

We used before the variational method for solving the variational problems. We proposed the existence of the solution *u* of the given problem. Then we considered a function  where *σ* is a number, and *h* was an arbitrary function, which was equal to zero for all points, where the unknown function was given. So the function  was admissible. Then we determined the function  of one variable *σ* . It has the minimum at the point . Therefore, the derivative of *f* at zero is equal to 0.

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| **Question**: Is the standard method applicable for our problem? |

This method is not applicable for Problem 8.1 because the function  is not satisfying the isoperimetric equality (8.3) for the general case. Therefore, we try to choose more difficult variation. Consider the function



where  and  are numbers, and functions  and  satisfy the boundary conditions

 (8.4)

So the function  satisfies the equality (8.2). However, we can guarantee the realization of the equality (8.3). Determine the functions of two variables

** **

We have the problem of the minimization the function on the set of pairs which satisfies the equality**** The pair  is the solution of this problem because of the equalities

** **

**8.3. Lagrange multipliers method for the function minimization**

Consider the differentiable functions 

**Problem 8.2**. *Find the vector* , *which satisfies the equalities*

 (8.5)

*and minimizes the function f*0*.*

We will use Implicit function theorem for solving this problem.

**Implicit function theorem**. *Let us have the continuously differential vector-function*  *such that*  *and the matrix*  *is not degenerate. Then for any t close enough to t\* there exists a vector xt such that F*(*xt*,*t*) = 0.

Return to Problem 8.2. Determine the function



where .

**Definition 8.1**. *The function L is called* ***Lagrange function****; and the elements*  *are called* ***Lagrange multipliers****.*

We solve Problem 8.2 on the base of the following result

**Theorem 8.1**. *Let the point*  *be the solution of the given problem. Then there exists a non-zero vector*  such that

 (8.6)

**Proof**. The equality (8.6) has the completely form



It can be interpreted as a system of linear algebraic equations with respect to the number . Suppose the statement of the theorem is false. Then the previous system has only zero solution, and the matrix  is not degenerate.

Determine the vector-function



where  Consider the system of nonlinear algebraic equations

 (8.7)

Obviously  Find the derivatives



Then the matrix  is equal to .

Therefore the matrix  is not degenerate too. Using the Implicit Function Theorem the system (8.7) has a solution  for small enough positive value *t* such that  So we have the equalities



Therefore there exists a point  such that  with value



because the number *t* is positive. Hence the point *x*\* is not solution of the given problem. So our supposition about linear independence of the  are false. Then we have the equalities (8.6). The proof of the theorem is complete.

**Example 8.1**. ***The rectangle with given perimeter and maximal area****.* Let the rectangle has the sides  Its area is equal to  and the perimeter is  So we have the problem of the minimization of the function **** with additional condition  ****,whereis **** UsingLagrange multipliers method we determine Lagrange function

****

Then we have the equalities

****

So  Put is to the equality **;** we have **** Then we find Hence the rectangle with given perimeter and maximal area is the square.

Здесь можно расписать условие экстремума, раскрыв L показать, что можно после деления на λ1 получить аналогичные соотношения с множителями Лагранжа 1 и λ=λ2/λ1.

**8.3. Lagrange multiplier method for the minimization problem with isoperimetric condition**.

Using Lagrange multiplier method determine Lagrange function

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where the number *λ* is Lagrange multiplier. Then we have necessary conditions of minimum

 (8.7)

Determine the function

 (8.8)

Then we can find Lagrange function

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Determine its partial derivatives

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We obtain

****

where

** **

After integration by parts we get

****

Using equalities (8.6), we have



So we obtain



This equality is true for all functions , which satisfy the conditions (8.4). Using the Basic Lemma of Variations Calculus we get

 (8.9)

It is Euler equation for Problem 8.1.

**Theorem 8.1**. *The solution of Problem* 8.1 *satisfies the equality* (8.9).

We analyze now the properties of the condition (8.9).

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| **Question**: Is it sufficient for finding the solution of the problem? |

The equality (8.9) with the function *Н*,determined by the formula (8.8) is second order differential equation. Its general solutions depends from two arbitrary constants and Lagrange multiplier *λ*. We have two boundary conditions and isoperimetric conditions for finding these three unknown values. So we have the sufficient information for solving the problem in principle.

|  |
| --- |
| **Question**: What is the practical solving of the problem? |

We can find the functional dependence

 (8.10)

from second order differential equation (8.9) with known function Ф. We put it to the boundary conditions (8.2). So we obtain two equalities.

 (8.11)

 (8.12)

Then we put the function *u* from (8.10) to isoperimetric condition (8.3). So we have the additional equality

 (8.13)

Hence we have the system of three nonlinear algebraic equations (8.11) – (8.113) with respect to the unknown three values . We find it from this system. Then the formula (8.10) gives the solution of the initial problem.

The algorithm of solving of Lagrange Problem with isoperimetric conditions is given in Table 8.1.

Table. 8.1. The algorithm of solving of Lagrange Problem with isoperimetric conditions

|  |  |
| --- | --- |
| **step** | **Action** |
| 1 | Determine the functions *F* and *G* and the constants   for the concrete Problem 8.1. |
| 2 | Determine the function *H* by the formula (8.8). |
| 3 | Determine Euler equation (8.9). |
| 4 | Find the general solution of Euler equation;  determine the function Ф of the formula (8.10). |
| 5 | Put the general solution to the system (8.11) – (8.13). |
| 6 | Find the solution  of the system  of the nonlinear equations (8.11) – (8.13). |
| 7 | Put the values  to the formula (8.10) with the concrete function Ф; the result is the required function. |
| 8 | Calculate the value of the corresponding value of the integral *I*. |

**Remark 8.1**. Of course, we have necessary conditions of the minimum. Hence, the solution of Problem 8.1 satisfies the equality (8.9). However, its solution can be non-optimal for the initial extremum problem.

We use Lagrange multipliers methods for solving a concrete Problem 8.1.

**8.3. A spectrum problem**

Consider the functional

 (8.14)

with boundary conditions

 (8.15)

and isoperimetric condition

 (8.16)

**Problem 8.3**. *Find the function v*, *which satisfies the boundary conditions* (8.16), *the isoperimetric condition* (8.15), *and minimizes the functional* (8.14)*.*

We prove this problem with using the obtained algorithm.

1. Transform this problem to Problem 8.1. Determine the functions



and the constants



2. Determine the function *H* from the formula (8.8)



3. Find the derivatives



Then we get Euler equation

 (8.17)

We will analyze it with the boundary conditions

 (8.18)

and the isoperimetric condition

 (8.19)

Hence we have the problem of finding the function  and Lagrange multiplier *λ* from the second order differential equation (8.17) with two boundary conditions (8.18) and the additional isoperimetric condition (8.19).

4. The equation (8.17) can have different forms of the general solution. It depends from the sign of the number *λ*. Determine its sign. Multiply the equality (8.17) by *u*. After integration we have

 (8.20)

We transform the first integral by means of the integration by parts. We get



because of the conditions (8.16).

The second integral in the equality (8.20) is equal to 1. So this equality can be transformed to

. (8.21)

Therefore, the number *λ* is not positive. If it equal to zero, then  because of the equality (8.21). Then the function *u* is constant. Therefore, we have  by the boundary conditions. However, the equality (8.19) is false in this case. Hence, the number *λ* is negative.

The general solution of the equation (8.14) for a negative *λ* is

 (8.22)

where  are arbitrary constants. We obtain the equality (8.10) for our case. The function Ф is the right side of the last equality.

5. We put the value *u* from the formula (8.21) to the equalities (8.18), (8.19) for finding the constants .

6. Using the first equality (8.18), we get

.

By second condition (8.18), we obtain



If  then  because of the formula (8.22). But it contradicts the equality (8.19). So we have



This is the equation with respect to *λ*. It has the solutions

**** (8.23)

Hence there exist a lot of values of Lagrange multiplier *λ* such that the problem (8.17), (8.18) has non-zero solutions

, (8.24)

where  is an arbitrary constant. This value can be chosen such that the function  satisfies the equality (8.19). We have



Then we obtain



7. Now we determine the functions

. (8.25)

Hence the system (8.17) – (8.19) has the infinite set of the solutions  and . It is definite by the formula (8.23), (8.25).

8. The solution of the problem satisfies the system (8.17) – (8.19). Using equalities (8.14) and (8.19), we find

.

Its minimal value can be the solution of Problem 8.3. So we determine . Thus, our problem has the unique solution



**Remark 8.2**. The problem of finding non-zero solutions of the system (8.17), (8.18) is called *Sturm – Liouville Problem*. The corresponding values  and functions  are called the *eigenvalues* and *eigenfunctions* of this problem. However, the functions (8.25) only satisfy the isoperimetric conditions (8.19). Note that it is orthonormal system. The set of values  is called the *spectrum* of the problem (8.17), (8.18)

**Remark 8.3**. The system (8.17) – (8.19) has non-unique solution . However, the first of them only minimize our functional. Hence, we have the necessary but not sufficient conditions of the minimum.

**Remark 8.4**. Find the physical sense of the considered example. Let us consider the movement of the spring on the time interval [0,*Т*]. This phenomenon can be described by the distance  between the state of the spring in the time *t* and its state of the equilibrium. Calculate the kinetic and potential energies of the spring. The kinetic energy is  where *m* is the mass of the spring and *v* is its velocity. The velocity is the derivative of the way *у*; so  Then we calculate  If the kinetic energy is a constant, then the kinetic energy of the system during the given time interval is the product *KT.* In reality, *K* depends from time. Therefore, we obtain the integral



The potential energy *U* is the product of the force *F* and the way *y*. By *Hooke’s law* the force of the elasticity is  where *k* is the coefficient of the elasticity. Therefore, we have the formula  If the potential energy is a constant, then its value during the given time interval is the product *UT.* In our case we get



Let the initial and the final state of the system are given



We can consider the problem of the minimization of the kinetic energy *I* in the case of the fixed potential energy *J*\*. We solved before the partial case of this problem with following values of the parameters:



**8.4. The problem with many isoperimetric conditions**

We consider the integral

 (8.26)

with boundary conditions

 (8.27)

and many isoperimetric conditions

 (8.28)

**Problem 8.3**. *Find the function v*, *which satisfies the boundary conditions* (8.27), *the isoperimetric conditions* (8.28), *and minimizes the functional* (8.26)*.*

|  |
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| **Question**: How we can solve this problem? |

Using Lagrange multipliers method, we determine the function

 (8.29)

where the numbers(namely the components of the vector *λ*), is called *Lagrange multipliers.* It is true the following result.

**Theorem 8.2**. *The solution of Problem* 8.4 *satisfies Euler equation*

 (8.30)

Hence the function  and *n-*degree vector *λ* are unknown. It can be founded from the second order differential equation (8.30) with two boundary conditions (8.27), and *n* equalities (8.28).

### Outcome

* The problem of the minimization of the function with equalities conditions can be solved by means of Lagrange multipliers method; the relevant necessary conditions of the minimum corresponds the stationary condition with respect to Lagrange function.
* The minimization problem with isoperimetric condition can be solved by means of Lagrange multipliers method.
* The necessary conditions of minimum for the problem with isoperimetric condition consist of second order differential Euler equation with two given boundary conditions and isoperimetric condition with respect to unknown function and Lagrange multiplier.
* A spectrum problem is in application of this theory.
* The problem of finding the minimal kinetic energy of the spring oscillation with given potential energy can interpreted as the spectrum problem.
* If we have many isoperimetric conditions, we can use Lagrange multipliers method; besides the quantity of Lagrange multipliers is equal to the quantity of the isoperimetric condition.

**Task. Variational problem with isoperimetric condition**

Consider the problem of the minimization of the integral



with boundary conditions

 (\*)

or

 (\*\*)

and isoperimetric condition



Table. 8.2. The values of the parameters.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary conditions | *A* | *b* | *M* |
| 1 | \* | 0 | 2π | 2 |
| 2 | \*\* | 0 | π | 1 |
| 3 | \* | 0 | 2π | 1 |
| 4 | \*\* | 0 | π | 2 |
| 5 | \* | 0 | π | 2 |
| 6 | \*\* | 0 | π | 3 |
| 7 | \* | 0 | π | 3 |
| 8 | \*\* | 0 | 2π | 1 |
| 9 | \* | 0 | 2π | 3 |
| 10 | \*\* | -π | π | 1 |
| 11 | \* | -π | 0 | 1 |
| 12 | \*\* | -π | π | 2 |
| 13 | \* | -π | 0 | 2 |
| 14 | \*\* | 0 | 2π | 1 |
| 15 | \* | -π | 0 | 3 |
| 16 | \*\* | 0 | 2π | 2 |
| 17 | \* | -π | π | 1 |
| 18 | \*\* | 0 | 2π | 3 |

Steps of the task:

1. Give the concrete problem statement.
2. Write the Euler equation.
3. Verify the sign of the Lagrange multiplier with using multiplication of the Euler equation by unknown function and integration.
4. Find the general solution of the Euler equation; it depends from two constants and the Lagrange multiplier.
5. Using given boundary conditions and isoperimetric condition find three unknown constants.
6. Find the set of the solutions of the conditions of the extremum.
7. Calculate the value of the given integral for all solution of the conditions of the extremum.
8. Chose the minimum of these values; find the solution of the problem.

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**Next step**

We have the standard method for solving the problems of minimization for the different integral functionals. We consider before minimization problems with given boundary conditions. We have already considered the case of the additional isoperimetric conditions. However there exist minimization problems with additional constraints for the values of the unknown function at the each points of its domain. We will try to extend Lagrange multipliers method to these problems.

Здесь сразу рассмотреть многомерный случай и согласовать индексы без G и F

Не забыть про лямбда-нуль

В задании убрать интервалы п 2п